EXERCISES

1. Let $f, g\colon X \to \mathbb{C} \setminus \{0\}$ be such continuous maps that $|f(x) - g(x)| < |f(x)|$ for all $x \in X$. Show that f and g are homotopic.

We define

$$H : X \times \underbrace{I}_{[0,1]} \longrightarrow \mathbb{C} \setminus \{0\}$$

$$H(x,t) = t\,g(x) + (1-t)f(x)$$

H is a linear homotopy; we must make sure that there is not t for which $H(x,t) = 0$.

$$|\,t\,g(x) + (1-t)f(x)| = |\,f(x) - t\,(f(x) - g(x))|$$

reverse triangle inequality

$$\geq |f(x)| - t\,|f(x) - g(x)|$$

$$\geq |f(x)| - t\,|f(x)|$$

$$\geq (1-t)\,|f(x)|$$

When $t \neq 1$, $(1-t)\,|f(x)| \neq 0$.

When $t = 1$, $H(x,1) = g(x) \neq 0$.

$\Rightarrow f, g$ are homotopic.

1. Let $(A_\bullet, d_A)$ and $(B_\bullet, d_B)$ be two chain complexes and $f : A_\bullet \to B_\bullet$ a chain map. Define the mapping cylinder $(Z(f)_\bullet, d_Z)$ by

$$Z(f)_i := A_i \oplus A_{i-1} \oplus B_i$$

with the differential given by

$$d_Z(a', a'', b) = (d_A(a') + a'', -d_A(a''), -f(a'') + d_B(b)) \quad \forall a' \in A_i, a'' \in A_{i-1}, b \in B_i.$$

I.e. in matrix from we can write: $d_Z = \begin{pmatrix} d_A & Id & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}$

a) **[3 Points]** Show that $(Z(f)_\bullet, d_Z)$ is a chain complex.

b) **[5 Points]** Consider the maps

$$\xi : B_\bullet \to Z(f)_\bullet : b \mapsto (0, 0, b),$$
$$\eta : Z(f)_\bullet \to B_\bullet : (a', a'', b) \mapsto f(a') + b.$$

Show that ξ and η are chain maps.

c) **[4 Points]** Show that ξ is a chain homotopy equivalence between $B_\bullet$ and $Z(f)_\bullet$.

ⓐ $Z(f)_i$ are abelian for all i.

$$\cdots \to A_i \oplus A_{i-1} \oplus B_i \xrightarrow{(d_Z)_i} A_{i-1} \oplus A_{i-2} \oplus B_{i-1} \xrightarrow{(d_Z)_{i-1}} \cdots$$

We must show that

$$(d_Z)_{i-1} \circ (d_Z)_i = 0.$$

$$\begin{pmatrix} (d_A)_{i-1} & id & 0 \\ 0 & -(d_A)_{i-2} & 0 \\ 0 & -f_{i-2} & (d_B)_{i-1} \end{pmatrix} \begin{pmatrix} (d_A)_i & id & 0 \\ 0 & -(d_A)_{i-1} & 0 \\ 0 & -f_{i-1} & (d_B)_i \end{pmatrix}$$

$$= \begin{pmatrix} (d_A)_{i-1} \circ (d_A)_i & (d_A)_{i-1} - (d_A)_{i-1} & 0 \\ 0 & (d_A)_{i-2} \circ (d_A)_{i-1} & 0 \\ 0 & f_{i-2}(d_A)_{i-1} - (d_B)_{i-1} f_{i-1} & (d_B)_{i-1} \circ (d_B)_i \end{pmatrix}$$

since the diagram on the left commutes, this equals 0

$$\begin{tikzcd} A_{i-1} \arrow[r, "(d_A)_{i-1}"] \arrow[d, "f_{i-1}"] & A_{i-2} \arrow[d, "f_{i-2}"] \\ B_{i-1} \arrow[r, "(d_B)_{i-1}"'] & B_{i-2} \end{tikzcd}$$

(b)

$$\begin{tikzcd} A_i \oplus A_{i-1} \oplus B_i \arrow[r, "(d_Z)_i"] \arrow[d, bend left, "\zeta_i"] & A_{i-1} \oplus A_{i-2} \oplus B_{i-1} \arrow[d, bend left, "\zeta_{i-1}"] \\ B_i \arrow[u, bend left, "\xi_i"] \arrow[r, "(d_B)_i"'] & B_{i-1} \arrow[u, bend left, "\xi_{i-1}"] \end{tikzcd}$$

We must show that:

(1) $(d_B)_i \circ \zeta_i = \zeta_{i-1} \circ (d_Z)_i$ (ζ is a chain map)

(2) $\xi_{i-1} \circ (d_B)_i = (d_Z)_i \circ \xi_i$

① $\zeta_{i-1} \circ (d_Z)_i (a', a'', b)$

$$= \zeta_{i-1}\left((d_A)_i(a') + a'', \; -(d_A)_{i-1}(a''), \; -f_{i-1}(a'') + (d_B)_i(b)\right)$$

$$= f_{i-1}\left((d_A)_i(a') + a''\right) - f_{i-1}(a'') + (d_B)_i(b)$$

$$= f_{i-1} \circ (d_A)_i(a') + (d_B)_i(b)$$

$$= (d_B)_i \circ f_i(a') + (d_B)_i(b)$$

$$= (d_B)_i\left(f_i(a') + b\right)$$

$$= (d_B)_i \circ \zeta_i(a', a'', b)$$

ζ is a chain map

② $(d_Z)_i \circ \xi_i (b) = \begin{pmatrix} (d_A)_i & id & 0 \\ 0 & -(d_A)_{i-1} & 0 \\ 0 & -f_{i-1} & (d_B)_i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$

$$= \left(0, 0, (d_B)_i(b)\right)^T = \xi_{i-1} \circ (d_B)_i(b)$$

(c) $\eta_i \circ \xi_i(b) = \eta_i \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} = b \Rightarrow \eta \circ \xi = id$

$\xi_i \circ \eta_i(a', a'', b) = \xi_i(f(a')+b) = \begin{pmatrix} 0 \\ 0 \\ f(a')+b \end{pmatrix}$

$\xi \circ \eta - id\,(a', a'', b) = \begin{pmatrix} -a' \\ -a'' \\ f(a') \end{pmatrix}$

$= \begin{pmatrix} -id & 0 & 0 \\ 0 & -id & 0 \\ f & 0 & 0 \end{pmatrix}$

Need to find $h : Z_i \to Z_{i+1}$

$h \circ d_Z - d_Z \circ h = \xi \circ \eta - id$

Set $h = \begin{pmatrix} 0 & 0 & 0 \\ -id & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$h \circ d_Z - d_Z \circ h = \begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (d_A)_i & \mathrm{id} & 0 \\ 0 & -(d_A)_{i-1} & 0 \\ 0 & -f_{i-1} & (d_B)_i \end{pmatrix}$$

$$- \begin{pmatrix} (d_A)_{i+1} & \mathrm{id} & 0 \\ 0 & -(d_A)_i & 0 \\ 0 & -f_i & (d_B)_{i+1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -(d_A)_i & -\mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -\mathrm{id} & 0 & 0 \\ (d_A)_i & 0 & 0 \\ f_i & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\mathrm{id} & 0 & 0 \\ 0 & -\mathrm{id} & 0 \\ -f_i & 0 & 0 \end{pmatrix} = g \circ \eta - \mathrm{id}$$